# Lecture 20

In general, $f_n \to f$ a.e. $\not\Rightarrow$ $f_n \to f$ in measure.

<u>Ex 2</u>. On $\mathbb{R}$, let $f_n = \chi_{(n,n+1)}$. Then $f_n \to 0$ pointwise, but $\{x : |f_n(x)| > \frac{1}{2}\} = (n, n+1)$, which has measure 1. Thus $f_n \not\to 0$ in measure.

However, on a <u>finite</u> measure space we have the stronger implication:

<u>Egoroff's Thm</u> Let $\{f_n\}$, $f$ be measurable, $f_n \to f$ $\mu$-a.e., and assume $\mu(X) < \infty$. Then, $\forall \varepsilon > 0$ $\exists E \in \mathcal{M}$ s.t. $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $E^c$.

<u>Rem.</u> Recall uniform convergence $\Rightarrow$ $\forall \varepsilon > 0$ $\exists N$ s.t. $\{x : |f - f_n| \geq \varepsilon\} = \emptyset$, for $n \geq N$.

Thus, if $f_n \to f$ a.e., by Egoroff, $\forall \varepsilon > 0$ $\exists N$ s.t. $\{x : |f(x) - f_n(x)| > \varepsilon\} \subseteq E$ for $n \geq N$. $\Rightarrow$ $f_n \to f$ in measure.

Pf of ET. WLOG. $f_n \to f$ everywhere (as the nullset on which $f_n$ may not converge to $f$ can be included in $E$).

Pick $\varepsilon > 0$ and consider the double seq.

$$E_{n,k} = \bigcup_{m=n}^{\infty} \{x : |f(x) - f_m(x)| > 1/k\}.$$

For fixed $k$, $E_{n,k} \searrow$ and $\bigcap_{n=1}^{\infty} E_{n,k} = \emptyset$ since $f_n(x) \to f(x)$, $\forall x$. Since $\mu(E_{1,k}) < \infty$ (by $\mu(X) < \infty$), cont. from above $\Rightarrow$ $\lim_{n\to\infty} \mu(E_{n,k}) = 0$. $\Rightarrow$ $\exists n_k$ s.t. $\mu(E_{n_k,k}) < \varepsilon 2^{-k}$.

Let $E = \bigcup_{k=1}^{\infty} E_{n_k,k}$ $\Rightarrow$

$$\mu(E) \leq \sum_{k=1}^{\infty} 2^{-k} \varepsilon = \varepsilon .$$

Moreover, on $E^c = \bigcap_{k=1}^{\infty} E_{n_k,k}^c$, we have

$$\left| f(x) - f_n(x) \right| \leq \frac{1}{k}, \quad n \geq n_k$$

$\Rightarrow$ $f_n \to f$ uniformly on $E^c$ as claimed.